Definice: Nechť f je T-periodická funkce, která je integrabilní na intervalu $[0,T]$.

Její **Fourierovu řadu** definujeme jako $\frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right]$, kde $\omega = \frac{2\pi}{T}$ je její frekvence,

$$a_k = \frac{2}{T} \int\limits_0^T f(t) \cos(k\omega t)\, dt, \quad \text{for} \quad k \in \mathbb{N}_0, \qquad b_k = \frac{2}{T} \int\limits_0^T f(t) \sin(k\omega t)\, dt, \quad \text{for} \quad k \in \mathbb{N}.$$

(i) Pokud f je lichá, pak $a_k = 0$ a $b_k = \frac{4}{T} \int\limits_0^{T/2} f(t) \sin(k\omega t)\, dt$.

(ii) Pokud f je sudá, pak $b_k = 0$ a $a_k = \frac{4}{T} \int\limits_0^{T/2} f(t) \cos(k\omega t)\, dt$.

Jordanovo kritérium: Nechť f je T-periodická funkce, která je po částech spojitá na nějakém intervalu I délky T. Předpokládejme, že její derivace f' je po částech spojitá na I.

Nechť $f \sim \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right]$. Pak pro každé $t \in \mathbb{R}$ platí

$$\lim_{N\to\infty} \left(\frac{a_0}{2} + \sum_{k=1}^{N} \left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] \right) = \tfrac{1}{2}[f(t^-) + f(t^+)].$$

Pokud je f navíc spojitá $\mathbb{R}$, pak $\frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right]$ konverguje k f *stejnoměrně.*

Mějme funkci

t nebo x

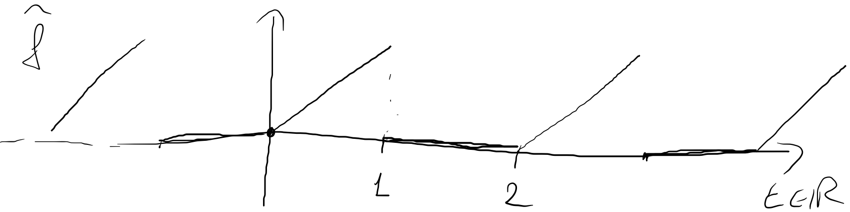
$$f(t) = \begin{cases} t & , t \in [0,1), \\ 0 & , t \in [1,2). \end{cases}$$

$T = 2, \quad \omega = \frac{2\pi}{T} = \pi$

Určete Fourierovu řadu

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos k\pi t + b_k \sin k\pi t \right)$$

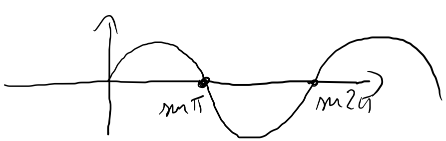
$$a_k = \frac{2}{T} \int_0^T f(t) \cos k\pi t \, dt, \quad b_k = \frac{2}{T} \int_0^T f(t) \sin k\pi t$$



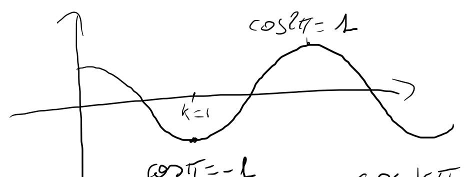
$$a_0 = \frac{2}{2} \int_0^2 f(t)\, dt = \int_0^1 t \, dt + \int_1^2 0 \, dt = \frac{1}{2} \qquad \frac{a_0}{2} = \frac{1}{4}$$

$$a_k = \int_0^2 f(t) \cos k\pi t \, dt = \int_0^1 t \cos k\pi t \, dt = \begin{vmatrix} t & \cos k\pi t \\ 1 & \frac{\sin k\pi t}{k\pi} \end{vmatrix} = \left[\frac{t \sin k\pi t}{k\pi} \right]_0^1 - \int_0^1 \frac{\sin k\pi t}{k\pi} \, dt =$$

$$= \left[t \frac{\sin k\pi t}{k\pi} + \frac{\cos k\pi t}{k^2 \pi^2} \right]_0^1 = \left(\frac{\sin k\pi}{k\pi} + \frac{\cos k\pi}{k^2\pi^2} - \frac{1}{k^2\pi^2} \right) = \frac{(-1)^k - 1}{k^2 \pi^2}$$



$\sin k\pi = 0$

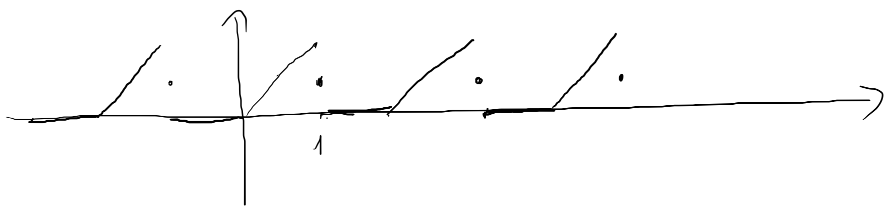


$\cos k\pi = (-1)^k$

$$b_k = \int_0^2 f(t)\sin k\pi t\,dt = \int_0^1 t\sin k\pi t\,dt = \begin{vmatrix} t & \sin k\pi t \\ 1 & -\dfrac{\cos k\pi t}{k\pi} \end{vmatrix}$$

$$= \left[-\frac{t\cos k\pi t}{k\pi} + \frac{\sin k\pi t}{k^2\pi^2}\right]_0^1 = -\frac{\cos k\pi}{k\pi} + \cancel{\frac{\sin k\pi}{k^2\pi^2}}^{\,0} - 0 = \frac{(-1)^{k+1}}{k\pi}$$

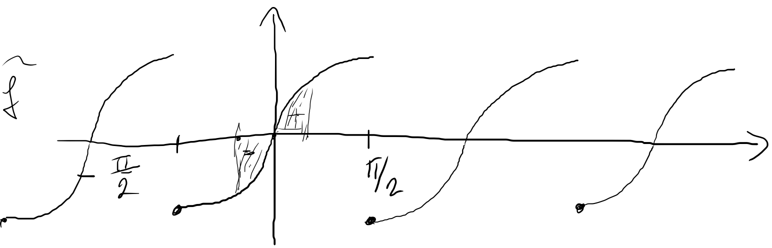
$$\hat{f} \sim \frac{1}{4} + \sum_{k=1}^{\infty}\left[\frac{(-1)^k - 1}{k^2\pi^2}\cos k\pi t + \frac{(-1)^{k+1}}{k\pi}\sin k\pi t\right]$$



Určete sinovou Fourierovu řadu příslušného periodického rozšíření funkce $f(t)=\sin t$, $0 \le t < \frac{\pi}{2}$. Určete funkci, ke které tato Fourierova řada konverguje.

$0 \le t < L$

lichá na $(-L, L)$



$f(t) = \sin t \quad -\pi/2 \le t < \pi/2$

sinová Fourierova řada pro funkce $f(t)=\sin t$ $0<t<\pi/2$

je Fourierova řada pro funkce $f(t)=\sin t$ $-\pi/2 \le t \le \pi/2$ (lichá)

$$a_k = \frac{2}{T}\int_0^T f(t)\cos k\omega t\,dt \qquad b_k = \frac{2}{T}\int_0^T f(t)\sin k\omega t\,dt$$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{f(t)\cos k\omega t}_{\text{lichá}}\,dt = 0$$

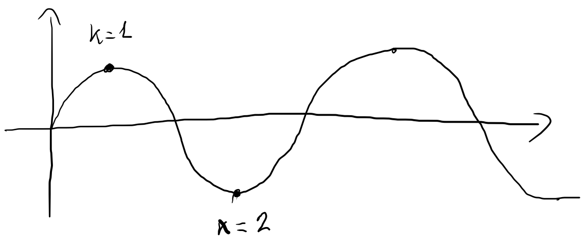
$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{f(t)\sin k\omega t}_{\text{sudá}}\,dt = 2\cdot\frac{2}{T}\int_0^{T/2} f(t)\sin k\omega t\,dt$$

$T = \pi \qquad \omega = \frac{2\pi}{T} = 2$

$$b_k = \frac{4}{\pi}\int_0^{\pi/2} \sin t \sin 2kt\,dt = \frac{2}{\pi}\int_0^{\pi/2}\left[\cos(2k-1)t - \cos(2k+1)t\right]dt = \frac{2}{\pi}\left[\frac{\sin(2k-1)t}{2k-1} - \frac{\sin(2k+1)t}{2k+1}\right]_0^{\pi/2}$$

$$= \frac{2}{\pi}\left[\frac{\sin(2k-1)\pi/2}{2k-1} - \frac{\sin(2k+1)\pi/2}{2k+1}\right]$$

$$b_k = \frac{2}{\pi}\left[\frac{\sin(2k-1)\pi/2}{2k-1} - \frac{\sin(2k+1)\pi/2}{2k+1}\right] = \frac{2}{\pi}\left[\frac{(-1)^{k+1}}{2k-1} - \frac{(-1)^{k}}{2k+1}\right] =$$



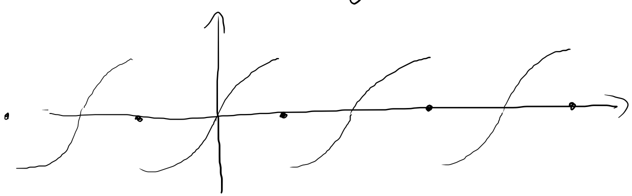
nebo $k=1$

$$= \frac{2}{\pi}\,\frac{(-1)^k\left[-2k-\cancel{1}-2k+\cancel{1}\right]}{4k^2-1} = \frac{8k(-1)^{k+1}}{\pi(4k^2-1)}$$

$$\hat{f} \sim \sum_{k=1}^{\infty} \frac{8k(-1)^{k+1}}{\pi(4k^2-1)} \sin 2kt = \frac{8}{3\pi}\sin 2t - \frac{16}{15\pi}\sin 4t + \frac{24}{35\pi}\sin 6t - \dots$$

$k=1$

$\downarrow$



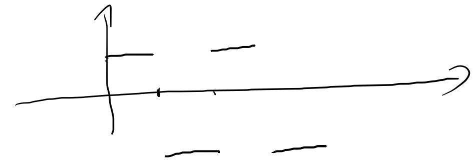
$$\cos x \cos y = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right)$$

$$\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$

$$\sin x \cos y = \frac{1}{2}\left(\sin(x+y) - \sin(x-y)\right)$$

Mějme funkci

$$f(t) = \begin{cases} 1 & , \ t \in [0,1), \\ -2 & , \ t \in [1,2). \end{cases}$$



Určete Fourierovu řadu příslušného periodického rozšíření funkce f.

- $f(t) = \max\{\sin t, 0\}, \quad t \in \langle 0, 2\pi)$

- $f(t) = \max\{\cos t, 0\}, \quad t \in \langle 0, 2\pi)$

sudá

Určete

- (a) Fourierovu řadu
- (b) sinovou Fourierovu řadu
- (c) kosinovou Fourierovu řadu

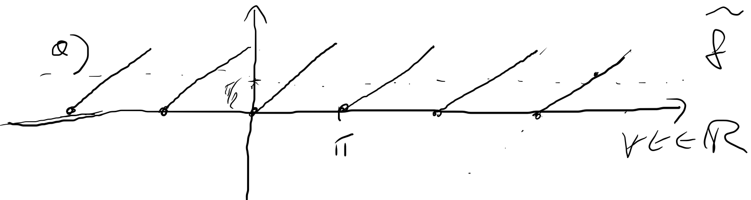
Mějme funkci

$f(t) = t,\ t \in (0, \pi)$

$\cos 2k\pi = 1$

$\sin 2k\pi = 0$

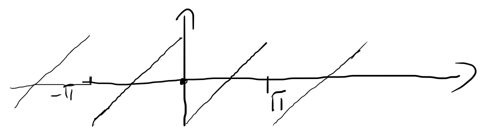
a)



$\tilde{f}$ $T = \pi$ $\omega = \frac{2\pi}{T} = 2$

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt = \frac{2}{\pi}\int_0^\pi t\,dt = \frac{2}{\pi}\,\frac{\pi^2}{2} = \pi \qquad \frac{a_0}{2} = \pi/2$$

$a_k = 0$ $\hat{\hat{f}} = \tilde{f} - \pi/2$



$\hat{\hat{f}}$ je liché

pro $\hat{\hat{f}}$ $a_k = 0$ $k \geq 1$

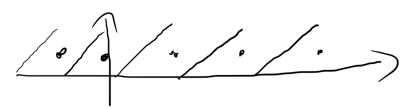
$\tilde{f} = \hat{\hat{f}} + \pi/2$ pro $\tilde{f}$ $a_k = 0,\ k \geq 1$

$$b_k = \frac{2}{T}\int_0^T f(t)\sin k\omega t\,dt = \frac{2}{\pi}\int_0^\pi t\sin 2kt\,dt = \begin{vmatrix} t & \sin 2kt \\ 1 & -\frac{\cos 2kt}{2k} \end{vmatrix} =$$

$$= \frac{2}{\pi}\left[-\frac{t\cos 2kt}{2k} + \frac{\sin 2kt}{4k^2}\right]_0^\pi = \frac{2}{\pi}\left[\frac{-\pi\cos 2k\pi}{2k} + \frac{\sin 2k\pi}{4k^2} - 0\right] = -\frac{1}{k}$$

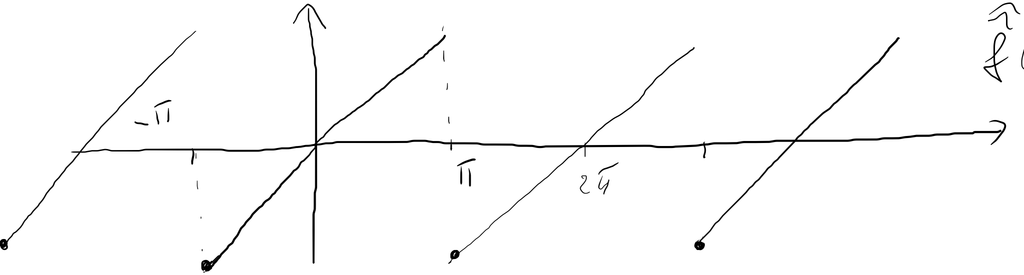
$$\tilde{f} \sim \frac{\pi}{2} + \sum_{k=1}^{\infty}\left(-\frac{1}{k}\sin 2kt\right)$$

$\forall t = k\pi$



$f(t) = t, \quad t \in \langle 0, \pi) \quad \langle 0, l)$

$\hat{f} = t, \quad t \in \langle -\pi, \pi \rangle \quad \langle -l, l \rangle$



$\hat{f}(t)$

$\hat{f}(25) = ?$

$\hat{f}$ je liché

Fou. řada pro $\hat{f}$ $a_k = 0$

$T = 2\pi, \quad \omega = \frac{2\pi}{2\pi} = 1$

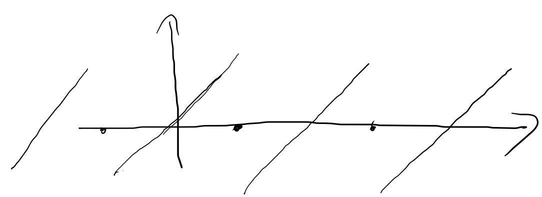
$$b_k = \frac{2}{T}\int_0^T f(t)\sin k\omega t\,dt = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{f(t)\sin}_{\text{sudá}} k\omega t\,dt = \frac{4}{T}\int_0^{T/2} f(t)\sin k\omega t\,dt$$

$$b_k = \frac{2}{\pi}\int_0^{\pi} t\sin kt\,dt = \frac{2}{\pi}\left[-\frac{t\cos kt}{k} + \frac{\sin kt}{k^2}\right]_0^{\pi} = \frac{2}{\pi}\left[-\frac{\pi\cos k\pi}{k} + \frac{\cancelto{0}{\sin k\pi}}{k^2}\right] =$$

$$= \frac{2(-1)^{k+1}}{k}$$

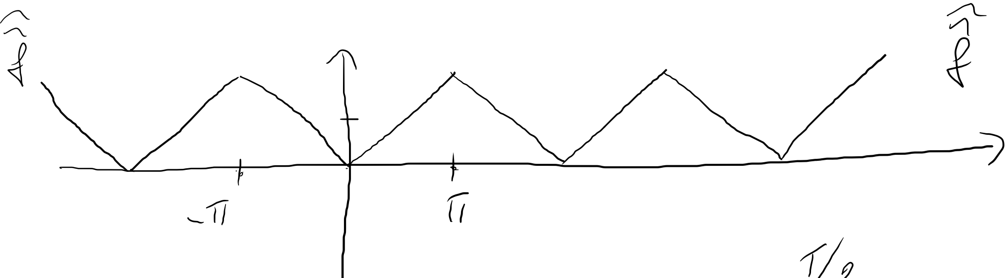
$$\left| \begin{array}{ll} t & \sin kt \\ 1 & -\frac{\cos kt}{k} \end{array} \right|$$

$$\sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kt$$



$$f(t) = t, \quad t \in \langle 0, \pi) \;\to\; \tilde f(t) = |t|, \quad t \in \langle -\pi, \pi)$$

Kosinová Fourierova řada:



$\tilde f$ je sudá $\qquad \underline{b_k = 0}$ $\qquad T = 2\pi$, $\quad \omega = \frac{2\pi}{T} = 1$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} t\,dt = \pi$$

$$a_k = \frac{2}{T}\int_0^{T} \underbrace{\underbrace{f(t)\cos k\omega t}_{\text{sudá} \cdot \text{sudé}}}_{\text{sudá}}\,dt = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{f(t)\cos k\omega t}_{\text{sudá}}\,dt = \frac{4}{T}\int_0^{T/2} f(t)\cos k\omega t\,dt$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} t\cos kt\,dt = \frac{2}{\pi}\left[\frac{t\sin kt}{k} + \frac{\cos kt}{k^2}\right]_0^{\pi} = \frac{2}{\pi}\left[\cancel{\frac{\pi\sin k\pi}{k}}^{\,0} + \frac{\cos k\pi}{k^2} - \frac{1}{k^2}\right]$$

$$\begin{vmatrix} t & \cos kt \\ 1 & \frac{\sin kt}{k} \end{vmatrix}$$

$$= \frac{2}{\pi}\,\frac{(-1)^k - 1}{k^2} = \begin{cases} 0 & k \text{ sudé} \\ -\frac{4}{\pi k^2}, & k \text{ liché} \end{cases}$$

$$\hat{\tilde f} = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{2}{\pi}\,\frac{(-1)^k - 1}{k^2}\cos kt = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-4}{\pi(2n-1)^2}\cos(2n-1)t =$$

$$k = 2n-1 \qquad = \frac{\pi}{2} - \frac{4}{\pi}\cos t - \frac{4}{9\pi}\cos 3t - \frac{4}{25\pi}\cos 5t - \dots$$

DÚ (cvič. 13)

$$\sum_{n=0}^{\infty}\left[2^n+(-1)^n\right] n x^n = \sum_{n=1}^{\infty}\left[2^n+(-1)^n\right] n x^n \qquad x_0 = 0$$

$$\lim_{n\to\infty} \frac{\left|2^{n+1}+(-1)^{n+1}\right| (n+1)\, |x|^{n+1}}{\left|2^{n}+(-1)^{n}\right| (n)\, |x|^{n}} = \lim_{n\to\infty} \frac{2^{n+1}\left(1+\overset{0}{\cancel{\left(\frac{-1}{2}\right)^{n+1}}}\right)}{2^{n}\left(1+\underset{0}{\cancel{\left(-\frac{1}{2}\right)^{n}}}\right)} \cdot \overset{1}{\cancel{\frac{n+1}{n}}} \cdot |x| = 2\,|x| < 1$$

$|x| < \frac{1}{2}$, $R = \frac{1}{2}$

pro $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ řada je abs. konv.

$$\sum_{n=1}^{\infty}\left[2^n+(-1)^n\right] n x^n = \sum_{n=1}^{\infty} n\,(2x)^n + \sum_{n=1}^{\infty} n\,(-x)^n \overset{\star}{=} \frac{2x}{(1-2x)^2} + \frac{-x}{(1+x)^2}$$

$$\star \quad \sum_{n=1}^{\infty} n y^n = y \sum_{n=1}^{\infty} n y^{n-1} = y \sum_{n=1}^{\infty}\left[y^n\right]' = y\left[\sum_{n=1}^{\infty} y^n\right]' = y\left[\frac{1}{1-y} - 1\right]' = \frac{y}{(1-y)^2}, \quad |y| < 1$$